

# An algebra generated by two sets of mutually orthogonal idempotents

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## Abstract

For a field  $\mathbb{F}$  and an integer  $d \geq 1$ , we consider the universal associative  $\mathbb{F}$ -algebra  $A$  generated by two sets of  $d+1$  mutually orthogonal idempotents. We display four bases for the  $\mathbb{F}$ -vector space  $A$  that we find attractive. We determine how these bases are related to each other. We describe how the multiplication in  $A$  looks with respect to our bases. Using our bases we obtain an infinite nested sequence of 2-sided ideals for  $A$ . Using our bases we obtain an infinite exact sequence involving a certain  $\mathbb{F}$ -linear map  $\partial : A \rightarrow A$ . We obtain several results concerning the kernel of  $\partial$ ; for instance this kernel is a subalgebra of  $A$  that is free of rank  $d$ .

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## 1 The algebra $A$

Throughout the paper  $\mathbb{F}$  denotes a field. All unadorned tensor products are meant to be over  $\mathbb{F}$ . An algebra is meant to be associative and have a 1.

We now introduce our topic.

**Definition 1.1** Let  $d$  denote a positive integer. Let  $A = A(d, \mathbb{F})$  denote the  $\mathbb{F}$ -algebra defined by generators  $\{e_i\}_{i=0}^d$ ,  $\{e_i^*\}_{i=0}^d$  and the following relations:

$$e_i e_j = \delta_{i,j} e_i, \quad e_i^* e_j^* = \delta_{i,j} e_i^*, \quad (0 \leq i, j \leq d), \quad (1)$$

$$1 = \sum_{i=0}^d e_i, \quad 1 = \sum_{i=0}^d e_i^*. \quad (2)$$

Here  $\delta_{i,j}$  denotes the Kronecker delta.

**Definition 1.2** Referring to Definition 1.1, we call  $\{e_i\}_{i=0}^d$  and  $\{e_i^*\}_{i=0}^d$  the *idempotent generators* for  $A$ . We say that the  $\{e_i^*\}_{i=0}^d$  are *starred* and the  $\{e_i\}_{i=0}^d$  are *nonstarred*.

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We now briefly explain how  $A$  can be viewed as a coproduct in the sense of Bergman [1, 2]. As we will see in Theorem 2.5, the elements  $\{e_i\}_{i=0}^d$  are linearly independent in  $A$  and hence form a basis for a subalgebra of  $A$  denoted  $A_1$ . Similarly the elements  $\{e_i^*\}_{i=0}^d$  form a basis for a subalgebra of  $A$  denoted  $A_1^*$ . By construction  $A$  is generated by  $A_1, A_1^*$ . The  $\mathbb{F}$ -algebras  $A_1$  and  $A_1^*$  are each isomorphic to a direct sum of  $d + 1$  copies of  $\mathbb{F}$ . The elements  $\{e_i\}_{i=0}^d$  (resp.  $\{e_i^*\}_{i=0}^d$ ) are the primitive idempotents of  $A_1$  (resp.  $A_1^*$ ). Since no relation in (1), (2) involves both  $A_1$  and  $A_1^*$ , the algebra  $A$  is the coproduct of  $A_1$  and  $A_1^*$  in the sense of Bergman [1, Section 1]. As part of his comprehensive study of coproducts, Bergman determined the units and zero-divisors in  $A$  [1, Corollary 2.16].

Our goal in this article is to describe four bases for  $A$  that we find attractive. We determine how these bases are related to each other. We describe how the multiplication in  $A$  looks with respect to these bases. Using our bases we obtain an infinite nested sequence of 2-sided ideals for  $A$ . Using our bases we obtain an infinite exact sequence involving a certain  $\mathbb{F}$ -linear map  $\partial : A \rightarrow A$ . We show that the kernel  $F$  of  $\partial$  is a subalgebra of  $A$  that is free of rank  $d$ . We show that  $F$  is generated by the elements  $\{e_i - e_i^*\}_{i=1}^d$ . We show that each of the  $\mathbb{F}$ -linear maps

$$\begin{array}{ccc} F \otimes A_1 & \rightarrow & A \\ u \otimes v & \mapsto & uv \end{array} \qquad \begin{array}{ccc} F \otimes A_1^* & \rightarrow & A \\ u \otimes v & \mapsto & uv \end{array}$$

is an isomorphism of  $\mathbb{F}$ -vector spaces. We will define our bases after a few comments.

The following three lemmas are about symmetries of  $A$ ; their proofs are routine and left to the reader.

**Lemma 1.3** *There exists a unique  $\mathbb{F}$ -algebra automorphism of  $A$  that sends*

$$e_i \mapsto e_i^*, \qquad e_i^* \mapsto e_i$$

*for  $0 \leq i \leq d$ . Denoting this automorphism by  $*$  we have  $x^{**} = x$  for all  $x \in A$ .*

By an  $\mathbb{F}$ -algebra antiautomorphism of  $A$  we mean an isomorphism of  $\mathbb{F}$ -vector spaces  $\rho : A \rightarrow A$  such that  $(xy)^\rho = y^\rho x^\rho$  for all  $x, y \in A$ .

**Lemma 1.4** *There exists a unique  $\mathbb{F}$ -algebra antiautomorphism  $\dagger$  of  $A$  that fixes each idempotent generator. We have  $x^{\dagger\dagger} = x$  for all  $x \in A$ .*

**Lemma 1.5** *The maps  $*$  and  $\dagger$  commute.*

Let  $X$  denote a subset of  $A$ . By the *relatives* of  $X$  we mean the subsets  $X, X^*, X^\dagger, X^{*\dagger}$ .

## 2 Four bases for the vector space $A$

In this section we display four bases for the  $\mathbb{F}$ -vector space  $A$ .

**Definition 2.1** A pair of idempotent generators for  $A$  is called *alternating* whenever one of them is starred and the other is nonstarred. For an integer  $n \geq 1$ , by a *word of length  $n$*  in  $A$  we mean a product  $g_1 g_2 \cdots g_n$  such that  $\{g_i\}_{i=1}^n$  are idempotent generators for  $A$  and  $g_{i-1}, g_i$  are alternating for  $2 \leq i \leq n$ . The word  $g_1 g_2 \cdots g_n$  is said to *begin* with  $g_1$  and *end* with  $g_n$ .

**Example 2.2** For  $d = 2$  we display the words in  $A$  that have length 3 and begin with  $e_0$ .

$$\begin{array}{lll} e_0 e_0^* e_0, & e_0 e_0^* e_1, & e_0 e_0^* e_2, \\ e_0 e_1^* e_0, & e_0 e_1^* e_1, & e_0 e_1^* e_2, \\ e_0 e_2^* e_0, & e_0 e_2^* e_1, & e_0 e_2^* e_2. \end{array}$$

**Definition 2.3** For an idempotent generator  $e_i$  or  $e_i^*$  we call  $i$  the *index* of the generator. A word  $g_1 g_2 \cdots g_n$  in  $A$  is called *nonrepeating* (or *NR*) whenever  $g_{j-1}, g_j$  do not have the same index for  $2 \leq j \leq n$ .

**Example 2.4** For  $d = 2$  we display the NR words in  $A$  that have length 3 and begin with  $e_0$ .

$$e_0 e_1^* e_0, \quad e_0 e_1^* e_2, \quad e_0 e_2^* e_0, \quad e_0 e_2^* e_1.$$

**Theorem 2.5** *Each of the following is a basis for the  $\mathbb{F}$ -vector space  $A$ :*

- (i) *The set of NR words in  $A$  that end with a nonstarred element.*
- (ii) *The set of NR words in  $A$  that end with a starred element.*
- (iii) *The set of NR words in  $A$  that begin with a nonstarred element.*
- (iv) *The set of NR words in  $A$  that begin with a starred element.*

*Proof:* (i) Let  $S$  denote the set of NR words in  $A$  that end with a nonstarred element. We first show that  $S$  spans  $A$ . Let  $A'$  denote the subspace of  $A$  spanned by  $S$ . To obtain  $A' = A$  it suffices to show that  $A'$  is a left ideal of  $A$  that contains 1. To show that  $A'$  is a left ideal of  $A$ , it suffices to show that  $e_i A' \subseteq A'$  and  $e_i^* A' \subseteq A'$  for  $0 \leq i \leq d$ . For a word  $w = g_1 g_2 \cdots g_n$  in  $S$  and  $0 \leq i \leq d$  we show that each of  $e_i w$ ,  $e_i^* w$  is contained in  $A'$ . Let  $j$  denote the index of  $g_1$ . Invoking (2) we may assume without loss that  $i \neq j$ . First assume  $n$  is odd, so that  $g_1 = e_j$ . Since  $e_i e_j = 0$  we have  $e_i w = 0$ , so  $e_i w \in A'$ . Also  $e_i^* w = e_i^* g_1 g_2 \cdots g_n$  is a word in  $S$ , so  $e_i^* w \in A'$ . Next assume  $n$  is even, so that  $g_1 = e_j^*$ . Then  $e_i w = e_i g_1 g_2 \cdots g_n$  is a word in  $S$ , so  $e_i w \in A'$ . Since  $e_i^* e_j^* = 0$  we have  $e_i^* w = 0$ , so  $e_i^* w \in A'$ . We have shown  $A'$  is a left ideal of  $A$ . The ideal  $A'$  contains 1, since  $e_i \in S$  for  $0 \leq i \leq d$  and  $1 = \sum_{i=0}^d e_i$ . We have shown  $A'$  is a left ideal of  $A$  that contains 1, so  $A' = A$ . Therefore  $S$  spans  $A$ . Next we show that the elements of  $S$  are linearly independent. Let  $\mathcal{S}$  denote the set of sequences  $(r_1, r_2, \dots, r_n)$  such that (i)  $n$  is a positive integer; (ii) each of  $r_1, r_2, \dots, r_n$  is contained in the set  $\{0, 1, \dots, d\}$ ; (iii)  $r_{i-1} \neq r_i$  for  $2 \leq i \leq n$ . Let  $V$  denote the vector space over  $\mathbb{F}$  consisting of those formal linear combinations of  $\mathcal{S}$  that have finitely many nonzero coefficients. The set  $\mathcal{S}$  is a basis for  $V$ . For  $0 \leq i \leq d$  we define linear transformations  $E_i : V \rightarrow V$  and  $E_i^* : V \rightarrow V$ . To this end we give the actions of  $E_i$  and  $E_i^*$  on  $\mathcal{S}$ . Pick an element  $(r_1, r_2, \dots, r_n) \in \mathcal{S}$ . The actions of  $E_i$  and  $E_i^*$  on  $(r_1, r_2, \dots, r_n)$  are given in the table below.

| Case                         | $E_i.(r_1, \dots, r_n)$  | $E_i^*(r_1, \dots, r_n)$   |
|------------------------------|--|--|
| $r_1 = i, n \text{ odd}$     | $(r_1, \dots, r_n)$  | $(r_1, \dots, r_n) - \sum_{\substack{0 \leq j \leq d \\ j \neq i}} (j, r_1, \dots, r_n)$ |
| $r_1 \neq i, n \text{ odd}$  | 0  | $(i, r_1, \dots, r_n)$   |
| $r_1 = i, n \text{ even}$    | $(r_1, \dots, r_n) - \sum_{\substack{0 \leq j \leq d \\ j \neq i}} (j, r_1, \dots, r_n)$ | $(r_1, \dots, r_n)$  |
| $r_1 \neq i, n \text{ even}$ | $(i, r_1, \dots, r_n)$   | 0  |

Using the table,

$$E_i E_j = \delta_{i,j} E_i, \quad E_i^* E_j^* = \delta_{i,j} E_i^*, \quad (0 \leq i, j \leq d), \quad (3)$$

$$1 = \sum_{i=0}^d E_i, \quad 1 = \sum_{i=0}^d E_i^*. \quad (4)$$

Comparing (3), (4) with (1), (2) we find that  $V$  has an  $A$ -module structure such that  $e_i$  (resp.  $e_i^*$ ) acts on  $V$  as  $E_i$  (resp.  $E_i^*$ ) for  $0 \leq i \leq d$ . Define the element  $\Delta \in V$  by  $\Delta = \sum_{i=0}^d (i)$ , and consider the linear transformation  $\gamma : A \rightarrow V$  that sends  $x \mapsto x.\Delta$  for all  $x \in A$ . For each word  $w = g_1 g_2 \cdots g_n$  in  $S$  we find  $\gamma(w) = (\overline{g_1}, \overline{g_2}, \dots, \overline{g_n})$  where  $\overline{g}$  denotes the index of  $g$ . Thus the restriction of  $\gamma$  to  $S$  gives a bijection  $S \rightarrow \mathcal{S}$ . The elements of  $\mathcal{S}$  are linearly independent and  $\gamma$  is linear, so the elements of  $S$  are linearly independent. We have shown  $S$  is a basis for  $A$ .

(ii) Apply the automorphism  $*$  to the basis in (i) above.

(iii), (iv) Apply the antiautomorphism  $\dagger$  to the bases in (i), (ii) above.  $\square$

### 3 How the four bases for $A$ are related

In this section we obtain some identities that effectively give the transition matrix between any two bases from Theorem 2.5.

**Notation 3.1** Let  $w = g_1 g_2 \cdots g_n$  denote a word in  $A$ , with  $g_n$  nonstarred. We represent  $w$  by the sequence  $(r_1, r_2, \dots, r_n)$ , where  $r_j$  denotes the index of  $g_j$  for  $1 \leq j \leq n$ . We represent  $w^*$  by  $(r_1, r_2, \dots, r_n)^*$ .

**Example 3.2** We display some words in  $A$  along with their notation.

| word                  | notation         |
|-----------------------|------------------|
| $e_0 e_2^* e_1$       | $(0, 2, 1)$      |
| $e_1^* e_0 e_2^* e_1$ | $(1, 0, 2, 1)$   |
| $e_0^* e_2 e_1^*$     | $(0, 2, 1)^*$    |
| $e_1 e_0^* e_2 e_1^*$ | $(1, 0, 2, 1)^*$ |

The next result effectively gives the transition matrix between any two bases from Theorem 2.5.

**Theorem 3.3** *With reference to Notation 3.1, and for each basis vector  $(r_1, r_2, \dots, r_n)$  from Theorem 2.5(i), the element*

$$(r_1, r_2, \dots, r_n) + (-1)^n (r_1, r_2, \dots, r_n)^*$$

*is equal to*

$$\begin{aligned} \sum_{\substack{0 \leq j \leq d \\ j \neq r_1}} (j, r_1, r_2, \dots, r_n) &+ \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\substack{0 \leq j \leq d \\ j \neq r_\ell, j \neq r_{\ell+1}}} (r_1, r_2, \dots, r_\ell, j, r_{\ell+1}, \dots, r_n) \\ &+ (-1)^n \sum_{\substack{0 \leq j \leq d \\ j \neq r_n}} (r_1, r_2, \dots, r_n, j), \end{aligned}$$

*and also equal to*

$$\begin{aligned} \sum_{\substack{0 \leq j \leq d \\ j \neq r_n}} (r_1, r_2, \dots, r_n, j)^* &+ \sum_{\ell=1}^{n-1} (-1)^{n-\ell} \sum_{\substack{0 \leq j \leq d \\ j \neq r_\ell, j \neq r_{\ell+1}}} (r_1, r_2, \dots, r_\ell, j, r_{\ell+1}, \dots, r_n)^* \\ &+ (-1)^n \sum_{\substack{0 \leq j \leq d \\ j \neq r_1}} (j, r_1, r_2, \dots, r_n)^*. \end{aligned}$$

*Proof:* To obtain the first assertion, define

$$\phi_0 = \sum_{\substack{0 \leq j \leq d \\ j \neq r_1}} (j, r_1, r_2, \dots, r_n), \quad (5)$$

$$\phi_\ell = \sum_{\substack{0 \leq j \leq d \\ j \neq r_\ell, j \neq r_{\ell+1}}} (r_1, r_2, \dots, r_\ell, j, r_{\ell+1}, \dots, r_n) \quad (1 \leq \ell \leq n-1), \quad (6)$$

$$\phi_n = \sum_{\substack{0 \leq j \leq d \\ j \neq r_n}} (r_1, r_2, \dots, r_n, j). \quad (7)$$

Evaluating (5)–(7) using (2) we find

$$\phi_0 = (r_1, r_2, \dots, r_n) - (r_1, r_1, r_2, \dots, r_n), \quad (8)$$

$$\phi_\ell = -(r_1, r_2, \dots, r_\ell, r_\ell, r_{\ell+1}, \dots, r_n) - (r_1, r_2, \dots, r_\ell, r_{\ell+1}, r_{\ell+1}, \dots, r_n), \quad (9)$$

$$\phi_n = (r_1, r_2, \dots, r_n)^* - (r_1, r_2, \dots, r_n, r_n). \quad (10)$$

Combining (8)–(10) we obtain

$$\phi_0 + \sum_{\ell=1}^{n-1} (-1)^\ell \phi_\ell + (-1)^n \phi_n = (r_1, r_2, \dots, r_n) + (-1)^n (r_1, r_2, \dots, r_n)^*,$$

and the first assertion follows. The second assertion is similarly obtained.  $\square$

**Example 3.4** Assume  $d = 2$ . Then for  $n = 1$  and  $r_1 = 1$  the assertions of Theorem 3.3 become

$$\begin{aligned} e_1 - e_1^* &= e_0^* e_1 + e_2^* e_1 - e_1^* e_0 - e_1^* e_2 \\ &= e_1 e_0^* + e_1 e_2^* - e_0 e_1^* - e_2 e_1^*. \end{aligned}$$

For  $n = 2$  and  $(r_1, r_2) = (1, 0)$  the assertions of Theorem 3.3 become

$$\begin{aligned} e_1^* e_0 + e_1 e_0^* &= e_0 e_1^* e_0 + e_2 e_1^* e_0 - e_1 e_2^* e_0 + e_1 e_0^* e_1 + e_1 e_0^* e_2 \\ &= e_1^* e_0 e_1^* + e_1^* e_0 e_2^* - e_1^* e_2 e_0^* + e_0^* e_1 e_0^* + e_2^* e_1 e_0^*. \end{aligned}$$

## 4 The product of basis elements

Consider the basis for  $A$  from Theorem 2.5(i). We now take two elements from this basis, and write the product as a linear combination of elements from the basis.

**Theorem 4.1** *Let  $(r_1, r_2, \dots, r_n)$  and  $(r'_1, r'_2, \dots, r'_m)$  denote basis vectors from Theorem 2.5(i). Then the product*

$$(r_1, r_2, \dots, r_n) \cdot (r'_1, r'_2, \dots, r'_m) \tag{11}$$

*is the following linear combination of basis vectors from Theorem 2.5(i).*

(i) *Assume  $m$  is odd and  $r_n \neq r'_1$ . Then (11) is zero.*

(ii) *Assume  $m$  is odd and  $r_n = r'_1$ . Then (11) is equal to*

$$(r_1, r_2, \dots, r_n, r'_2, \dots, r'_m).$$

(iii) *Assume  $m$  is even and  $r_n \neq r'_1$ . Then (11) is equal to*

$$(r_1, r_2, \dots, r_n, r'_1, r'_2, \dots, r'_m).$$

(iv) *Assume  $m$  is even and  $r_n = r'_1$ . Then (11) is equal to*

$$\begin{aligned} &(-1)^{n+1} (r_1, r_2, \dots, r_n, r'_2, \dots, r'_m) + (-1)^n \sum_{\substack{0 \leq j \leq d \\ j \neq r_1}} (j, r_1, r_2, \dots, r_n, r'_2, \dots, r'_m) \\ &+ \sum_{\ell=1}^{n-1} (-1)^{n-\ell} \sum_{\substack{0 \leq j \leq d \\ j \neq r_\ell, j \neq r_{\ell+1}}} (r_1, r_2, \dots, r_\ell, j, r_{\ell+1}, \dots, r_n, r'_2, \dots, r'_m). \end{aligned}$$

*Proof:* (i)–(iii) Routine.

(iv) In line (11), evaluate  $(r_1, r_2, \dots, r_n)$  using the second identity in Theorem 3.3, and simplify the result.  $\square$

Now consider the basis for  $A$  from Theorem 2.5(ii), and the basis for  $A$  from Theorem 2.5(i). In the next result, we take an element from the first basis and an element from the second basis, and write the product as a linear combination of elements from the second basis.

**Theorem 4.2** *In the notation of Theorem 4.1, the product*

$$(r_1, r_2, \dots, r_n)^* \cdot (r'_1, r'_2, \dots, r'_m) \quad (12)$$

*is the following linear combination of basis vectors from Theorem 2.5(i).*

(i) *Assume  $m$  is even and  $r_n \neq r'_1$ . Then (12) is 0.*

(ii) *Assume  $m$  is even and  $r_n = r'_1$ . Then (12) is equal to*

$$(r_1, r_2, \dots, r_n, r'_2, \dots, r'_m).$$

(iii) *Assume  $m$  is odd and  $r_n \neq r'_1$ . Then (12) is equal to*

$$(r_1, r_2, \dots, r_n, r'_1, r'_2, \dots, r'_m).$$

(iv) *Assume  $m$  is odd and  $r_n = r'_1$ . Then (12) is equal to*

$$\begin{aligned} & (-1)^{n+1} (r_1, r_2, \dots, r_n, r'_2, \dots, r'_m) + (-1)^n \sum_{\substack{0 \leq j \leq d \\ j \neq r_1}} (j, r_1, r_2, \dots, r_n, r'_2, \dots, r'_m) \\ & + \sum_{\ell=1}^{n-1} (-1)^{n-\ell} \sum_{\substack{0 \leq j \leq d \\ j \neq r_\ell, j \neq r_{\ell+1}}} (r_1, r_2, \dots, r_\ell, j, r_{\ell+1}, \dots, r_n, r'_2, \dots, r'_m). \end{aligned}$$

*Proof:* (i)–(iii) Routine.

(iv) In line (12), evaluate  $(r_1, r_2, \dots, r_n)^*$  using the first identity in Theorem 3.3, and simplify the result.  $\square$

## 5 The subspaces $A_n$

In this section we introduce some subspaces  $A_n$  of  $A$ , and use them to interpret our results so far.

**Definition 5.1** For an integer  $n \geq 1$  let  $A_n$  denote the subspace of  $A$  spanned by the NR words that have length  $n$  and end with a nonstarred element.

**Lemma 5.2** *For  $n \geq 1$  we display a basis for each relative of  $A_n$ .*

| space            | basis  |
|------------------|--|
| $A_n$            | the NR words in $A$ that have length $n$ and end with a nonstarred element   |
| $A_n^*$          | the NR words in $A$ that have length $n$ and end with a starred element      |
| $A_n^\dagger$    | the NR words in $A$ that have length $n$ and begin with a nonstarred element |
| $A_n^{*\dagger}$ | the NR words in $A$ that have length $n$ and begin with a starred element    |

*Proof:* Immediate from Lemma 1.3, Lemma 1.4, and Theorem 2.5.  $\square$

**Lemma 5.3** *For  $n \geq 1$  each relative of  $A_n$  has dimension  $(d+1)d^{n-1}$ .*

*Proof:* Apply Lemma 5.2 and a routine counting argument.  $\square$

**Lemma 5.4** *The following (i), (ii) hold for all integers  $n \geq 1$ .*

(i) *Suppose  $n$  is even. Then  $A_n^{*\dagger} = A_n$  and  $A_n^* = A_n^\dagger$ .*

(ii) *Suppose  $n$  is odd. Then  $A_n^{*\dagger} = A_n^*$  and  $A_n^\dagger = A_n$ .*

*Proof:* Pick a word  $w$  in  $A$  of length  $n$ . If  $n$  is even, then  $w$  begins with a starred element if and only if  $w$  ends with a nonstarred element. If  $n$  is odd, then  $w$  begins with a starred element if and only if  $w$  ends with a starred element. The result follows.  $\square$

**Theorem 5.5** *Each of the following sums is direct.*

$$\begin{aligned} A &= \sum_{n=1}^{\infty} A_n, & A &= \sum_{n=1}^{\infty} A_n^*, \\ A &= \sum_{n=1}^{\infty} A_n^\dagger, & A &= \sum_{n=1}^{\infty} A_n^{*\dagger}. \end{aligned}$$

*Proof:* Combine Theorem 2.5 and Lemma 5.2.  $\square$

**Theorem 5.6** *For  $n \geq 1$  and  $x \in A_n$ ,*

$$x + (-1)^n x^* \in A_{n+1} \cap A_{n+1}^*.$$

*Proof:* By Definition 5.1 we may assume without loss that  $x$  is an NR word in  $A$  that has length  $n$  and ends with a nonstarred element. Now  $x + (-1)^n x^* \in A_{n+1}$  by the first assertion of Theorem 3.3, and  $x + (-1)^n x^* \in A_{n+1}^*$  by the second assertion of Theorem 3.3. The result follows.  $\square$

**Corollary 5.7** *For  $n \geq 1$ ,*

$$A_n + A_{n+1} \cap A_{n+1}^* = A_n^* + A_{n+1} \cap A_{n+1}^*.$$

*Proof:* This is a routine consequence of Theorem 5.6.  $\square$

For subsets  $X, Y$  of  $A$  let  $XY$  denote the subspace of  $A$  spanned by  $\{xy \mid x \in X, y \in Y\}$ .

**Theorem 5.8** *For positive integers  $n, m$  the products  $A_n A_m$  and  $A_n^* A_m$  are described as follows.*

(i) *Assume  $m$  is odd. Then*

$$A_n A_m \subseteq A_{n+m-1}, \quad A_n^* A_m \subseteq A_{n+m} + A_{n+m-1}. \quad (13)$$

(ii) *Assume  $m$  is even. Then*

$$A_n A_m \subseteq A_{n+m} + A_{n+m-1}, \quad A_n^* A_m \subseteq A_{n+m-1}. \quad (14)$$

*Proof:* In (13) and (14) the inclusions on the left follow from Theorem 4.1, and the inclusions on the right follow from Theorem 4.2.  $\square$

In Section 9 we will obtain a more detailed version of Theorem 5.8.

## 6 The ideals $A_{\geq n}$

Motivated by Corollary 5.7 and Theorem 5.8 we consider the following subspaces of  $A$ .

**Definition 6.1** For  $n \geq 1$  define

$$A_{\geq n} = A_n + A_{n+1} + \cdots$$

**Theorem 6.2** *For  $n \geq 1$  the space  $A_{\geq n}$  is a 2-sided ideal of  $A$ .*

*Proof:* This is a routine consequence of the inclusions on the left in (13), (14).  $\square$

**Theorem 6.3** *For  $n \geq 1$  we have*

$$A_{\geq n}^* = A_{\geq n}, \quad A_{\geq n}^\dagger = A_{\geq n}.$$

*Proof:* For  $m \geq 1$  we obtain  $A_m \subseteq A_m^* + A_{m+1}^*$  and  $A_m^* \subseteq A_m + A_{m+1}$  from Corollary 5.7. Therefore  $A_{\geq n}^* = A_{\geq n}$ . For  $m \geq 1$  the space  $A_m^\dagger$  is one of  $A_m$ ,  $A_m^*$  by Lemma 5.4, and each of  $A_m$ ,  $A_m^*$  is contained in  $A_m + A_{m+1}$ , so  $A_m^\dagger \subseteq A_m + A_{m+1}$ . In this inclusion we apply  $\dagger$  to each side and find  $A_m \subseteq A_m^\dagger + A_{m+1}^\dagger$ . Therefore  $A_{\geq n}^\dagger = A_{\geq n}$ .  $\square$

**Lemma 6.4** *For positive integers  $n, m$  the product  $A_{\geq n} A_{\geq m}$  is contained in  $A_{\geq n+m-1}$ .*

*Proof:* This follows from Definition 6.1 and the products on the left in (13), (14).  $\square$

## 7 The map $\partial : A \rightarrow A$

Motivated by Theorem 5.6 we consider the following map.

**Lemma 7.1** There exists a unique  $\mathbb{F}$ -linear transformation  $\partial : A \rightarrow A$  such that for  $n \geq 1$ ,

$$\partial(x) = x + (-1)^n x^* \quad (\forall x \in A_n). \quad (15)$$

*Proof:* By Theorem 5.5 the sum  $A = \sum_{n=1}^{\infty} A_n$  is direct.  $\square$

**Lemma 7.2** With reference to Lemma 7.1 we have  $\partial(A_n) \subseteq A_{n+1}$  for  $n \geq 1$ .

*Proof:* Immediate from Theorem 5.6.  $\square$

**Lemma 7.3** With reference to Lemma 7.1 the following (i), (ii) hold for all  $x \in A$ .

$$(i) \quad \partial(\partial(x)) = 0.$$

$$(ii) \quad \partial(x^*) = -(\partial(x))^*.$$

*Proof:* Without loss we may assume  $x \in A_n$  for some  $n \geq 1$ .

(i) Observe  $\partial(x) \in A_{n+1}$  by Lemma 7.2, so  $\partial(\partial(x)) = \partial(x) - (-1)^n (\partial(x))^*$  by (15). In line (15) we apply  $*$  to both sides and get  $(\partial(x))^* = (-1)^n \partial(x)$ . The result follows.

(ii) In line (15) we apply  $\partial$  to both sides and use (i) above to get  $\partial(x^*) = (-1)^{n-1} \partial(x)$ . We observed  $(\partial(x))^* = (-1)^n \partial(x)$  in the proof of part (i), so  $\partial(x^*) = -(\partial(x))^*$ .  $\square$

**Lemma 7.4** For  $n \geq 1$  the kernel of  $\partial$  on  $A_n$  is  $A_n \cap A_n^*$ .

*Proof:* For  $x \in A_n$  we show that  $\partial(x) = 0$  if and only if  $x \in A_n^*$ . First assume  $\partial(x) = 0$ . Then  $x^* = (-1)^{n-1} x$  by (15), so  $x \in A_n^*$ . To get the reverse implication, assume  $x \in A_n^*$  and note that  $x^* \in A_n$ . Now each of  $x, x^*$  is contained in  $A_n$ , so  $\partial(x) \in A_n$  in view of (15). But  $\partial(x) \in A_{n+1}$  by Lemma 7.2 and  $A_n \cap A_{n+1} = 0$  by Theorem 5.5 so  $\partial(x) = 0$ .  $\square$

Our next goal is to show that for  $n \geq 1$  the image of  $A_n$  under  $\partial$  is  $A_{n+1} \cap A_{n+1}^*$ . To this end it will be convenient to introduce some subspaces  ${}^+A_n$  and  ${}^0A_n$  of  $A_n$ .

**Definition 7.5** For  $n \geq 1$  let  ${}^+A_n$  (resp.  ${}^0A_n$ ) denote the subspace of  $A_n$  with a basis consisting of the NR words that have length  $n$ , and end with one of  $e_1, e_2, \dots, e_d$  (resp. end with  $e_0$ ).

**Example 7.6** Assume  $d = 2$ . The basis for  ${}^+A_3$  from Definition 7.5 is

$$\begin{array}{cccc} e_1 e_0^* e_1, & e_2 e_0^* e_1, & e_0 e_2^* e_1, & e_1 e_2^* e_1, \\ e_1 e_0^* e_2, & e_2 e_0^* e_2, & e_0 e_1^* e_2, & e_2 e_1^* e_2. \end{array}$$

The basis for  ${}^0A_3$  from Definition 7.5 is

$$e_0 e_1^* e_0, \quad e_2 e_1^* e_0, \quad e_0 e_2^* e_0, \quad e_1 e_2^* e_0.$$

**Lemma 7.7** For  $n \geq 1$ ,

- (i)  $A_n = {}^+A_n + {}^0A_n$  (direct sum).
- (ii) The dimension of  ${}^+A_n$  is  $d^n$ .
- (iii) The dimension of  ${}^0A_n$  is  $d^{n-1}$ .

*Proof:* Routine using Lemma 5.2 and Definition 7.5. □

**Definition 7.8** For  $n \geq 1$  we define an isomorphism of vector spaces  $\sigma : {}^+A_n \rightarrow {}^0A_{n+1}$ . To do this we give the action of  $\sigma$  on the basis for  ${}^+A_n$  from Definition 7.5. Let  $(r_1, r_2, \dots, r_n)$  denote an NR word in  $A$  such that  $r_n \neq 0$ . We define the image of this word under  $\sigma$  to be  $(r_1, r_2, \dots, r_n, 0)$ . Note that  $\sigma$  sends the above basis for  ${}^+A_n$  to the basis for  ${}^0A_{n+1}$  given in Definition 7.5. Therefore  $\sigma$  is an isomorphism of vector spaces.

**Lemma 7.9** For  $n \geq 1$  and  $x \in {}^+A_n$ ,

$$\sigma(x) = (-1)^n \partial(x) e_0. \quad (16)$$

*Proof:* Without loss we may assume that  $x$  is a vector in the basis for  ${}^+A_n$  given in Definition 7.5. Thus  $x$  is an NR word  $(r_1, r_2, \dots, r_n)$  such that  $r_n \neq 0$ . Observe that  $x e_0 = 0$  and  $x^* e_0 = (r_1, r_2, \dots, r_n, 0)$ . By this and (15) we find  $(-1)^n \partial(x) e_0$  is equal to  $(r_1, r_2, \dots, r_n, 0)$ , which is equal to  $\sigma(x)$  by Definition 7.8. The result follows. □

**Lemma 7.10** For  $n \geq 1$ ,

$$A_n = {}^+A_n + A_n \cap A_n^* \quad (\text{direct sum}). \quad (17)$$

Moreover the dimension of  $A_n \cap A_n^*$  is  $d^{n-1}$ .

*Proof:* We first show that the sum  ${}^+A_n + A_n \cap A_n^*$  is direct. By Lemma 7.9 and since  $\sigma : {}^+A_n \rightarrow {}^0A_{n+1}$  is a bijection, the restriction of  $\partial$  to  ${}^+A_n$  is injective. Therefore the kernel of  $\partial$  on  $A_n$  has zero intersection with  ${}^+A_n$ . This kernel is  $A_n \cap A_n^*$  by Lemma 7.4. Therefore  ${}^+A_n$  has zero intersection with  $A_n \cap A_n^*$  so the sum  ${}^+A_n + A_n \cap A_n^*$  is direct. Let  $k_n$  denote the dimension of  $A_n \cap A_n^*$ . By our comments so far, and given the dimensions of  $A_n$  and  ${}^+A_n$  from Lemma 5.3 and Lemma 7.7, respectively, we obtain  $k_n \leq d^{n-1}$ , with equality if and only if  $A_n = {}^+A_n + A_n \cap A_n^*$ . To finish the proof it suffices to show  $k_n = d^{n-1}$ . We do this by induction on  $n$ . First assume  $n = 1$ . We have  $k_1 \leq 1$  by our above remarks, and  $k_1 \geq 1$  since  $1 \in A_1 \cap A_1^*$  by (2). Therefore  $k_1 = 1$  as desired. Next assume  $n \geq 2$ . Let  $I_n$  denote the image of  $A_{n-1}$  under  $\partial$ . By linear algebra the dimension of  $I_n$  is equal to the dimension of  $A_{n-1}$  minus the dimension of the kernel of  $\partial$  on  $A_{n-1}$ . The dimension of  $A_{n-1}$  is  $d^{n-1} + d^{n-2}$ . The kernel of  $\partial$  on  $A_{n-1}$  is  $A_{n-1} \cap A_{n-1}^*$  so its dimension is  $k_{n-1}$ , which is  $d^{n-2}$  by induction. Therefore the dimension of  $I_n$  is  $d^{n-1}$ . By Theorem 5.6 and (15) we have  $I_n \subseteq A_n \cap A_n^*$ . In this inclusion we consider the dimensions and get  $d^{n-1} \leq k_n$ . We showed earlier that  $k_n \leq d^{n-1}$  so  $k_n = d^{n-1}$  as desired. The result follows. □

**Lemma 7.11** For  $n \geq 1$  the image of  $A_n$  under  $\partial$  is  $A_{n+1} \cap A_{n+1}^*$ .

*Proof:* Denote this image by  $I_{n+1}$ , and observe  $I_{n+1} \subseteq A_{n+1} \cap A_{n+1}^*$  by Theorem 5.6. To finish the proof we show that  $I_{n+1}$  and  $A_{n+1} \cap A_{n+1}^*$  have the same dimension. By Lemma 7.10 the dimension of  $A_{n+1} \cap A_{n+1}^*$  is  $d^n$ . By Lemma 7.4 and (17) the dimension of  $I_{n+1}$  is equal to the dimension of  ${}^+A_n$ , which is  $d^n$  by Lemma 7.7(ii). The result follows.  $\square$

**Lemma 7.12** We have  $A_1 \cap A_1^* = \mathbb{F}1$ .

*Proof:* Observe  $\mathbb{F}1 \subseteq A_1 \cap A_1^*$  by (2), and  $A_1 \cap A_1^*$  has dimension 1 by Lemma 7.10.  $\square$

**Definition 7.13** Let  $\iota : \mathbb{F} \rightarrow A$  denote  $\mathbb{F}$ -algebra homomorphism that sends  $a \mapsto a1$  for  $a \in \mathbb{F}$ . Note that  $\iota$  is an injection.

**Theorem 7.14** The sequence

$$\mathbb{F} \xrightarrow{\iota} A_1 \xrightarrow{\partial} A_2 \xrightarrow{\partial} A_3 \xrightarrow{\partial} \cdots$$

is exact in the sense of [3, p. 435].

*Proof:* This follows from Lemma 7.4, Lemma 7.11, and Lemma 7.12.  $\square$

We emphasize a few points for later use.

**Lemma 7.15** For  $n \geq 1$  the restriction of  $\partial$  to  ${}^+A_n$  is an isomorphism of vector spaces  ${}^+A_n \rightarrow A_{n+1} \cap A_{n+1}^*$ .

*Proof:* Combine Lemma 7.4, line (17), and Lemma 7.11.  $\square$

**Lemma 7.16** For  $n \geq 1$  and  $x \in A_n$  the following are equivalent:

- (i)  $x^* = (-1)^{n-1}x$ ;
- (ii)  $x \in A_n \cap A_n^*$ .

*Proof:* Combine (15) and Lemma 7.4.  $\square$

**Lemma 7.17** For  $n \geq 1$  the map  $\partial$  acts on  $A_n^*$  as follows.

$$\partial(y) = -y - (-1)^n y^* \quad (\forall y \in A_n^*).$$

*Proof:* Write  $x = y^*$ , so that  $x \in A_n$  and  $y = x^*$ . Now compute  $\partial(y)$  using Lemma 7.3(ii) and (15).  $\square$

## 8 A subalgebra of $A$

In this section we consider the sum

$$\sum_{n=0}^{\infty} (A_{n+1} \cap A_{n+1}^*). \quad (18)$$

We observe by Theorem 5.5 and Lemma 7.4 that (18) is the kernel of the map  $\partial : A \rightarrow A$ . We will show that (18) is a subalgebra of  $A$  that is free of rank  $d$ .

**Lemma 8.1** *For nonnegative integers  $n, m$  the following (i)–(iii) hold.*

- (i)  $(A_{n+1} \cap A_{n+1}^*)A_{m+1} \subseteq A_{n+m+1}$ ;
- (ii)  $(A_{n+1} \cap A_{n+1}^*)A_{m+1}^* \subseteq A_{n+m+1}^*$ ;
- (iii)  $(A_{n+1} \cap A_{n+1}^*)(A_{m+1} \cap A_{m+1}^*) \subseteq A_{n+m+1} \cap A_{n+m+1}^*$ .

*Proof:* (i) For  $x \in A_{n+1} \cap A_{n+1}^*$  and  $y \in A_{m+1}$  we show that  $xy \in A_{n+m+1}$ . First assume  $m$  is even. Using  $x \in A_{n+1}$  and  $y \in A_{m+1}$  and the inclusion on the left in (13), we obtain  $xy \in A_{n+m+1}$ . Next assume  $m$  is odd. Using  $x \in A_{n+1}^*$  and  $y \in A_{m+1}$  and the inclusion on the right in (14), we obtain  $xy \in A_{n+m+1}$ .

(ii) For  $x \in A_{n+1} \cap A_{n+1}^*$  and  $y \in A_{m+1}^*$  we show that  $xy \in A_{n+m+1}^*$ . Observe that  $x^* \in A_{n+1} \cap A_{n+1}^*$  and  $y^* \in A_{m+1}$  so  $x^*y^* \in A_{n+m+1}$  by (i) above. Applying  $*$  we find  $xy \in A_{n+m+1}^*$ .

(iii) Combine (i) and (ii) above. □

**Corollary 8.2** *The sum (18) is a subalgebra of  $A$ .*

*Proof:* The sum contains the identity 1 of  $A$  by Lemma 7.12. The sum is closed under multiplication by Lemma 8.1(iii). □

We will return to the subalgebra (18) after a few comments.

**Lemma 8.3** *For each basis vector  $(r_1, r_2, \dots, r_n)$  from Theorem 2.5(i), the element*

$$(r_1, r_2, \dots, r_n) + (-1)^n (r_1, r_2, \dots, r_n)^*$$

*is equal to*

$$(e_{r_1} - e_{r_1}^*)(e_{r_2} - e_{r_2}^*) \cdots (e_{r_n} - e_{r_n}^*)(-1)^{\lfloor n/2 \rfloor}. \quad (19)$$

*The expression  $\lfloor x \rfloor$  denotes the greatest integer less than or equal to  $x$ .*

*Proof:* Expand (19) into a sum of  $2^n$  terms. Simplify these terms using (1) and  $r_{i-1} \neq r_i$  for  $2 \leq i \leq n$ .  $\square$

Let  $F$  denote the  $\mathbb{F}$ -algebra defined by generators  $\{s_i\}_{i=1}^d$  and no relations. Thus  $F$  is the free  $\mathbb{F}$ -algebra of rank  $d$ . We call  $\{s_i\}_{i=1}^d$  the *standard generators* for  $F$ . We recall a few facts about  $F$ . For an integer  $n \geq 0$ , by a *word in  $F$  of length  $n$*  we mean a product  $y_1 y_2 \cdots y_n$  such that  $\{y_i\}_{i=1}^n$  are standard generators for  $F$ . We interpret the word of length 0 to be the identity of  $F$ . The  $\mathbb{F}$ -vector space  $F$  has a basis consisting of its words [3, p. 723]. For  $n \geq 0$  let  $F_n$  denote the subspace of  $F$  spanned by the words of length  $n$ . Note that  $F_n$  has dimension  $d^n$ . We have a direct sum  $F = \sum_{n=0}^{\infty} F_n$ , and  $F_r F_s = F_{r+s}$  for  $r, s \geq 0$ . We call  $F_n$  the  *$n$ th homogeneous component* of  $F$ .

**Theorem 8.4** *With the above notation, consider the  $\mathbb{F}$ -algebra homomorphism  $F \rightarrow A$  that sends  $s_i \mapsto e_i - e_i^*$  for  $1 \leq i \leq d$ . This map is an injection and its image is  $\sum_{n=0}^{\infty} (A_{n+1} \cap A_{n+1}^*)$ . Moreover for  $n \geq 0$  the image of  $F_n$  is  $A_{n+1} \cap A_{n+1}^*$ .*

*Proof:* Let  $\varepsilon : F \rightarrow A$  denote the homomorphism in question. We claim that for  $n \geq 0$  the restriction of  $\varepsilon$  to  $F_n$  is a bijection  $F_n \rightarrow A_{n+1} \cap A_{n+1}^*$ . To establish the claim we split the argument into three cases:  $n = 0$ ,  $n = 1$ , and  $n \geq 2$ . The claim holds for  $n = 0$  by Lemma 7.12 and since  $F_0 = \mathbb{F}1$ . To see that the claim holds for  $n = 1$ , note that  $F_1$  has a basis  $\{s_i\}_{i=1}^d$ . By Definition 7.5 the elements  $\{e_i\}_{i=1}^d$  form a basis for  ${}^+A_1$ , so  $\{\partial(e_i)\}_{i=1}^d$  is a basis for  $A_2 \cap A_2^*$  in view of Lemma 7.15. By Lemma 7.1 we have  $\partial(e_i) = e_i - e_i^*$  for  $1 \leq i \leq d$ . Therefore  $\{e_i - e_i^*\}_{i=1}^d$  is a basis for  $A_2 \cap A_2^*$ , and the claim follows for  $n = 1$ . We now show that the claim holds for  $n \geq 2$ . Using  $F_n = (F_1)^n$ ,  $\varepsilon(F_1) = A_2 \cap A_2^*$ , and Lemma 8.1(iii) we obtain  $\varepsilon(F_n) \subseteq A_{n+1} \cap A_{n+1}^*$ . To see the reverse inclusion, first note by Lemma 7.11 that any element in  $A_{n+1} \cap A_{n+1}^*$  can be written as  $\partial(x)$  for some  $x \in A_n$ . We show  $\partial(x) \in \varepsilon(F_n)$ . Without loss we may assume that  $x$  is a vector  $(r_1, r_2, \dots, r_n)$  in the basis for  $A_n$  from Lemma 5.2. Combining Lemma 7.1 and Lemma 8.3 we find that  $\partial(x)$  is equal to (19). In particular  $\partial(x) = \varepsilon(z_1)\varepsilon(z_2)\cdots\varepsilon(z_n)$  where  $z_i \in F_1$  for  $1 \leq i \leq n$ . Observe  $\partial(x) = \varepsilon(z_1 z_2 \cdots z_n)$  and  $z_1 z_2 \cdots z_n \in F_n$  so  $\partial(x) \in \varepsilon(F_n)$ . Therefore  $A_{n+1} \cap A_{n+1}^* \subseteq \varepsilon(F_n)$ . So far we have  $\varepsilon(F_n) = A_{n+1} \cap A_{n+1}^*$ . To show that the map  $F_n \rightarrow A_{n+1} \cap A_{n+1}^*$ ,  $x \mapsto \varepsilon(x)$  is a bijection, it suffices to show that  $F_n$  and  $A_{n+1} \cap A_{n+1}^*$  have the same dimension. We mentioned below Lemma 8.3 that  $F_n$  has dimension  $d^n$ . By the last line of Lemma 7.10 we find  $A_{n+1} \cap A_{n+1}^*$  also has dimension  $d^n$ . By these comments the map  $F_n \rightarrow A_{n+1} \cap A_{n+1}^*$ ,  $x \mapsto \varepsilon(x)$  is a bijection. The claim is now proved for  $n \geq 2$ . We have established the claim, and the result follows in view of the directness of the sum  $\sum_{n=0}^{\infty} A_{n+1} \cap A_{n+1}^*$ .  $\square$

For notational convenience let us identify the free algebra  $F$  from above Theorem 8.4 with the subalgebra (18) of  $A$ , via the injection from Theorem 8.4. Our next goal is to show that each of the  $\mathbb{F}$ -linear maps

$$\begin{array}{ll} F \otimes A_1 \rightarrow A & F \otimes A_1^* \rightarrow A \\ u \otimes v \mapsto uv & u \otimes v \mapsto uv \end{array}$$

is an isomorphism of  $\mathbb{F}$ -vector spaces. We need a lemma.

**Lemma 8.5** For positive integers  $n, m$  the  $\mathbb{F}$ -linear map

$$\begin{aligned} (A_n \cap A_n^*) \otimes A_m &\rightarrow A_{n+m-1} \\ u \otimes v &\mapsto uv \end{aligned}$$

is an isomorphism of  $\mathbb{F}$ -vector spaces.

*Proof:* Let  $\theta$  denote the map in question. To show that  $\theta$  is bijective, we show that the dimension of  $(A_n \cap A_n^*) \otimes A_m$  is equal to the dimension of  $A_{n+m-1}$ , and that  $\theta$  is surjective. The dimension of  $A_n \cap A_n^*$  is  $d^{n-1}$  by Lemma 7.10, and the dimension of  $A_m$  is  $(d+1)d^{m-1}$  by Lemma 5.3, so the dimension of  $(A_n \cap A_n^*) \otimes A_m$  is  $(d+1)d^{n+m-2}$ . The dimension of  $A_{n+m-1}$  is  $(d+1)d^{n+m-2}$  by Lemma 5.3. Therefore the dimensions of  $(A_n \cap A_n^*) \otimes A_m$  and  $A_{n+m-1}$  are the same. Next we show that  $\theta$  is surjective. First assume  $n = 1$ . Then  $\theta$  is surjective since  $1 \in A_1 \cap A_1^*$  by Lemma 7.12. Next assume  $n \geq 2$ . By Lemma 5.2 the space  $A_{n+m-1}$  has a basis consisting of the NR words in  $A$  that have length  $n+m-1$  and end with a nonstarred element. We show that each of these basis elements is in the image of  $\theta$ . Consider an NR word  $w = (r_1, r_2, \dots, r_{n+m-1})$ . Define  $u = (-1)^{(n-1)m} \partial(r_1, r_2, \dots, r_{n-1})$  and observe that  $u \in A_n \cap A_n^*$  by Lemma 7.11. Define  $v = (r_n, r_{n+1}, \dots, r_{n+m-1})$  and observe  $v \in A_m$ . One verifies  $w = uv$  by first using (15), and then Theorem 4.1(i), Theorem 4.2(iii) if  $m$  is odd and Theorem 4.1(iii), Theorem 4.2(i) if  $m$  is even. Therefore  $w$  is the image of  $u \otimes v$  under  $\theta$ . We have shown that  $\theta$  is surjective, and the result follows.  $\square$

**Theorem 8.6** Each of the  $\mathbb{F}$ -linear maps

$$\begin{aligned} F \otimes A_1 &\rightarrow A & F \otimes A_1^* &\rightarrow A \\ u \otimes v &\mapsto uv & u \otimes v &\mapsto uv \end{aligned}$$

is an isomorphism of  $\mathbb{F}$ -vector spaces.

*Proof:* Let  $\psi$  (resp.  $\xi$ ) denote the map on the left (resp. right). We first show that  $\psi$  is an isomorphism of  $\mathbb{F}$ -vector spaces. By construction the sum  $F = \sum_{n=1}^{\infty} A_n \cap A_n^*$  is direct. Therefore the sum

$$F \otimes A_1 = \sum_{n=1}^{\infty} (A_n \cap A_n^*) \otimes A_1$$

is direct. By Theorem 5.5 the sum  $A = \sum_{n=1}^{\infty} A_n$  is direct. For  $n \geq 1$  we apply Lemma 8.5 with  $m = 1$  and find that the map

$$\begin{aligned} (A_n \cap A_n^*) \otimes A_1 &\rightarrow A_n \\ u \otimes v &\mapsto uv \end{aligned}$$

is an isomorphism of  $\mathbb{F}$ -vector spaces. It follows that  $\psi$  is an isomorphism of  $\mathbb{F}$ -vector spaces. The map  $\xi$  is an isomorphism of  $\mathbb{F}$ -vector spaces since it is the composition

$$F \otimes A_1^* \xrightarrow{* \otimes *} F \otimes A_1 \xrightarrow{\psi} A \xrightarrow{*} A$$

and each composition factor is an isomorphism of  $\mathbb{F}$ -vector spaces.  $\square$

## 9 The subspaces $A_n$ revisited

In this section we present a more detailed version of Theorem 5.8. Let  $n, m$  denote positive integers. For  $m$  odd we consider the  $\mathbb{F}$ -linear maps

$$\begin{aligned} A_n \otimes A_m &\rightarrow A_{n+m-1} & A_n \otimes A_m &\rightarrow A_{n+m} + A_{n+m-1} \\ u \otimes v &\mapsto uv & u \otimes v &\mapsto u^*v \end{aligned}$$

and for  $m$  even we consider the  $\mathbb{F}$ -linear maps

$$\begin{aligned} A_n \otimes A_m &\rightarrow A_{n+m} + A_{n+m-1} & A_n \otimes A_m &\rightarrow A_{n+m-1} \\ u \otimes v &\mapsto uv & u \otimes v &\mapsto u^*v. \end{aligned}$$

**Definition 9.1** Let  $n, m$  denote positive integers.

- (i) Let  $\neq(A_n \otimes A_m)$  denote the subspace of  $A_n \otimes A_m$  that has a basis consisting of the elements  $u \otimes v$ , where  $u = (r_1, r_2, \dots, r_n)$  is an NR word in  $A_n$  and  $v = (r'_1, r'_2, \dots, r'_m)$  is an NR word in  $A_m$  such that  $r_n \neq r'_1$ .
- (ii) Let  $= (A_n \otimes A_m)$  denote the subspace of  $A_n \otimes A_m$  that has a basis consisting of the elements  $u \otimes v$ , where  $u = (r_1, r_2, \dots, r_n)$  is an NR word in  $A_n$  and  $v = (r'_1, r'_2, \dots, r'_m)$  is an NR word in  $A_m$  such that  $r_n = r'_1$ .

The following result is immediate from Definition 9.1.

**Lemma 9.2** *With reference to Definition 9.1,*

$$A_n \otimes A_m = \neq(A_n \otimes A_m) + = (A_n \otimes A_m) \quad (\text{direct sum}). \quad (20)$$

**Theorem 9.3** *For positive integers  $n, m$  the following (i), (ii) hold.*

- (i) *Assume  $m$  is odd. Then the  $\mathbb{F}$ -linear map*

$$\begin{aligned} A_n \otimes A_m &\rightarrow A_{n+m-1} \\ u \otimes v &\mapsto uv \end{aligned}$$

*is surjective with kernel  $\neq(A_n \otimes A_m)$ .*

- (ii) *Assume  $m$  is even. Then the  $\mathbb{F}$ -linear map*

$$\begin{aligned} A_n \otimes A_m &\rightarrow A_{n+m-1} \\ u \otimes v &\mapsto u^*v \end{aligned}$$

*is surjective with kernel  $\neq(A_n \otimes A_m)$ .*

*Proof:* (i) A basis for  $\neq(A_n \otimes A_m)$  is given in Definition 9.1(i). By Theorem 4.1(i) the map sends each element in this basis to zero. A basis for  $= (A_n \otimes A_m)$  is given in Definition 9.1(ii). By Theorem 4.1(ii) the map sends this basis to the basis for  $A_{n+m-1}$  given in Lemma 5.2. The result follows from these comments and Lemma 9.2.

- (ii) Similar to the proof of (i) above. □

**Lemma 9.4** *For positive integers  $n, m$  we have*

$$A_n \otimes A_m = \neq(A_n \otimes A_m) + (A_n \cap A_n^*) \otimes A_m \quad (\text{direct sum}). \quad (21)$$

*Proof:* For  $m$  odd the result follows from Lemma 8.5 and Theorem 9.3(i). For  $m$  even the result follows from Lemma 7.16, Lemma 8.5, and Theorem 9.3(ii).  $\square$

The following result will be helpful.

**Proposition 9.5** *For positive integers  $n, m$  the  $\mathbb{F}$ -linear map*

$$\begin{aligned} A_n \otimes A_m &\rightarrow A_{n+m} \\ u \otimes v &\mapsto \partial(u)v \end{aligned}$$

*is surjective with kernel  $(A_n \cap A_n^*) \otimes A_m$ .*

*Proof:* By Lemma 7.4 and Lemma 7.11, the map  $A_n \rightarrow A_{n+1} \cap A_{n+1}^*$ ,  $u \mapsto \partial(u)$  is surjective with kernel  $A_n \cap A_n^*$ . Therefore the map  $A_n \otimes A_m \rightarrow (A_{n+1} \cap A_{n+1}^*) \otimes A_m$ ,  $u \otimes v \mapsto \partial(u) \otimes v$  is surjective with kernel  $(A_n \cap A_n^*) \otimes A_m$ . By Lemma 8.5 the map  $(A_{n+1} \cap A_{n+1}^*) \otimes A_m \rightarrow A_{n+m}$ ,  $u \otimes v \mapsto uv$  is a bijection. Composing the two previous maps, we find that the map  $A_n \otimes A_m \rightarrow A_{n+m}$ ,  $u \otimes v \mapsto \partial(u)v$  is surjective with kernel  $(A_n \cap A_n^*) \otimes A_m$ .  $\square$

**Theorem 9.6** *For positive integers  $n, m$  the following (i), (ii) hold.*

(i) *Assume  $m$  is even. Then the  $\mathbb{F}$ -linear map*

$$\begin{aligned} A_n \otimes A_m &\rightarrow A_{n+m} + A_{n+m-1} \\ u \otimes v &\mapsto uv \end{aligned}$$

*is an isomorphism of  $\mathbb{F}$ -vector spaces. Under this map the preimage of  $A_{n+m}$  is  $\neq(A_n \otimes A_m)$  and the preimage of  $A_{n+m-1}$  is  $(A_n \cap A_n^*) \otimes A_m$ .*

(ii) *Assume  $m$  is odd. Then the  $\mathbb{F}$ -linear map*

$$\begin{aligned} A_n \otimes A_m &\rightarrow A_{n+m} + A_{n+m-1} \\ u \otimes v &\mapsto u^*v \end{aligned}$$

*is an isomorphism of  $\mathbb{F}$ -vector spaces. Under this map the preimage of  $A_{n+m}$  is  $\neq(A_n \otimes A_m)$  and the preimage of  $A_{n+m-1}$  is  $(A_n \cap A_n^*) \otimes A_m$ .*

*Proof:* For  $u \in A_n$  and  $v \in A_m$  we use  $\partial(u) = u + (-1)^n u^*$  to obtain

$$\partial(u)v = uv + (-1)^n u^*v. \quad (22)$$

(i) Denote the map by  $\eta$ . The restriction of  $\eta$  to  $\neq(A_n \otimes A_m)$  gives a bijection  $\neq(A_n \otimes A_m) \rightarrow A_{n+m}$  by Theorem 9.3(ii), Lemma 9.4, Proposition 9.5, and (22). The restriction of  $\eta$  to  $(A_n \cap A_n^*) \otimes A_m$  gives a bijection  $(A_n \cap A_n^*) \otimes A_m \rightarrow A_{n+m-1}$ , by Lemma 8.5. The result follows.

(ii) Denote the map by  $\zeta$ . The restriction of  $\zeta$  to  $\neq(A_n \otimes A_m)$  gives a bijection  $\neq(A_n \otimes A_m) \rightarrow A_{n+m}$  by Theorem 9.3(i), Lemma 9.4, Proposition 9.5, and (22). The restriction of  $\zeta$  to  $(A_n \cap A_n^*) \otimes A_m$  gives a bijection  $(A_n \cap A_n^*) \otimes A_m \rightarrow A_{n+m-1}$ , by Lemma 7.16 and Lemma 8.5. The result follows.  $\square$

## 10 The subspaces $A_{\leq n}$

In this last section we investigate the following subspaces of  $A$ .

**Definition 10.1** For all integers  $n \geq 1$  we define

$$A_{\leq n} = A_1 + A_2 + \cdots + A_n. \quad (23)$$

The following lemma is immediate from the construction.

**Lemma 10.2** For  $n \geq 1$  we display a basis for each relative of  $A_{\leq n}$ .

| space                   | basis  |
|-------------------------|--|
| $A_{\leq n}$            | the NR words in $A$ that have length at most $n$ and end with a nonstarred element   |
| $A_{\leq n}^*$          | the NR words in $A$ that have length at most $n$ and end with a starred element      |
| $A_{\leq n}^\dagger$    | the NR words in $A$ that have length at most $n$ and begin with a nonstarred element |
| $A_{\leq n}^{*\dagger}$ | the NR words in $A$ that have length at most $n$ and begin with a starred element    |

**Lemma 10.3** For  $n \geq 1$  the relatives of  $A_{\leq n}$  all have dimension  $(d+1)(1+d+d^2+\cdots+d^{n-1})$ .

*Proof:* By Theorem 5.5 and Definition 10.1, the dimension of  $A_{\leq n}$  is equal to the sum of the dimensions of  $A_1, A_2, \dots, A_n$ . The result follows from this and Lemma 5.3.  $\square$

In Lemma 10.2 we gave a basis for each relative of  $A_{\leq n}$ . In a moment we will display another basis. In order to motivate this new basis we first give a spanning set.

**Lemma 10.4** For  $n \geq 1$  we display a spanning set for each relative of  $A_{\leq n}$ .

| space                   | spanning set  |
|-------------------------|---|
| $A_{\leq n}$            | the words in $A$ that have length $n$ and end with a nonstarred element   |
| $A_{\leq n}^*$          | the words in $A$ that have length $n$ and end with a starred element      |
| $A_{\leq n}^\dagger$    | the words in $A$ that have length $n$ and begin with a nonstarred element |
| $A_{\leq n}^{*\dagger}$ | the words in $A$ that have length $n$ and begin with a starred element    |

*Proof:* Concerning the first row of the table, let  $S_n$  denote the subspace of  $A$  spanned by the words in  $A$  that have length  $n$  and end with a nonstarred element. We show  $S_n = A_{\leq n}$ . By construction  $S_n = \cdots A_1 A_1^* A_1$  ( $n$  factors). By Theorem 5.8 we have  $A_1 A_j \subseteq A_j + A_{j+1}$  and  $A_1^* A_j \subseteq A_j + A_{j+1}$  for  $1 \leq j \leq n-1$ . By this and induction on  $n$  we find  $S_n \subseteq A_{\leq n}$ . To get the reverse inclusion, note that for  $1 \leq j \leq n$  we have  $A_j \subseteq S_j$ , and also  $S_j \subseteq S_n$  since  $1 \in A_1$  and  $1 \in A_1^*$  by Lemma 7.12. We have verified the first row of the table. The remaining rows are similarly verified.  $\square$

For each spanning set in Lemma 10.4, the set is not a basis for  $n \geq 3$ , since the set has cardinality  $(d+1)^n$  and this number differs from the dimension given in Lemma 10.3. Our next goal is to obtain a subset of the spanning set that is a basis.

**Definition 10.5** A word  $g_1g_2\cdots g_n$  in  $A$  is called *repeating/nonrepeating* (or  $R/NR$ ) whenever for  $2 \leq j \leq n$ , if  $g_{j-1}, g_j$  have the same index then  $g_1, g_2, \dots, g_j$  all have the same index.

**Example 10.6** For  $d = 2$  we display the  $R/NR$  words in  $A$  that have length 3 and end with  $e_0$ .

$$\begin{array}{llll} e_0e_0^*e_0, & e_1e_1^*e_0, & e_2e_2^*e_0, & \\ e_0e_1^*e_0, & e_2e_1^*e_0, & e_0e_2^*e_0, & e_1e_2^*e_0. \end{array}$$

**Definition 10.7** A word in  $A$  is called *nonrepeating/repeating* (or  $NR/R$ ) whenever its image under  $\dagger$  is  $R/NR$ .

**Example 10.8** For  $d = 2$  we display the  $NR/R$  words in  $A$  that have length 3 and start with  $e_0$ .

$$\begin{array}{llll} e_0e_0^*e_0, & e_0e_1^*e_1, & e_0e_2^*e_2, & \\ e_0e_1^*e_0, & e_0e_1^*e_2, & e_0e_2^*e_0, & e_0e_2^*e_1. \end{array}$$

**Theorem 10.9** For  $n \geq 1$  we display a basis for each relative of  $A_{\leq n}$ .

| space                   | basis  |
|-------------------------|--|
| $A_{\leq n}$            | the $R/NR$ words in $A$ that have length $n$ and end with a nonstarred element   |
| $A_{\leq n}^*$          | the $R/NR$ words in $A$ that have length $n$ and end with a starred element      |
| $A_{\leq n}^\dagger$    | the $NR/R$ words in $A$ that have length $n$ and begin with a nonstarred element |
| $A_{\leq n}^{*\dagger}$ | the $NR/R$ words in $A$ that have length $n$ and begin with a starred element    |

*Proof:* Concerning the first row of the table, let  $(R/NR)_n$  denote the set of  $R/NR$  words in  $A$  that have length  $n$  and end with a nonstarred element. We show  $(R/NR)_n$  is a basis for  $A_{\leq n}$ . Let  $(NR)_n$  denote the basis for  $A_n$  given in Lemma 5.2. Let  $(NR)_{\leq n} = \cup_{j=1}^n (NR)_j$  denote the basis for  $A_{\leq n}$  given in Lemma 10.2. We now define a linear transformation  $f : A_{\leq n} \rightarrow A_{\leq n}$ . To this end we give the action of  $f$  on  $(NR)_j$  for  $1 \leq j \leq n$ . For a word  $(r_1, r_2, \dots, r_j)$  in  $(NR)_j$  we define its image under  $f$  to be  $(r_1, r_1, \dots, r_1, r_1, r_2, \dots, r_j)$  ( $n$  coordinates). This image is contained in  $A_{\leq n}$  by Lemma 10.4. By the construction  $f$  sends the basis  $(NR)_{\leq n}$  to the set  $(R/NR)_n$ . To show that  $(R/NR)_n$  is a basis for  $A_{\leq n}$  it suffices to show that  $f$  is a bijection. Using the data in Theorem 4.1 and Theorem 4.2, one finds  $(f - I)A_j \subseteq A_{j+1} + \cdots + A_n$  for  $1 \leq j \leq n$ , where  $I : A \rightarrow A$  is the identity map. Therefore, with respect to an appropriate ordering of the basis  $(NR)_{\leq n}$ , the matrix which represents  $f$  is lower triangular, with all diagonal entries 1. This matrix is nonsingular so  $f$  is invertible and hence a bijection. Therefore  $(R/NR)_n$  is a basis for  $A_{\leq n}$ . This yields the first row of the table. The remaining rows are similarly obtained.  $\square$

## References

- [1] G. Bergman. Modules over coproducts of rings. *Trans. Amer. Math. Soc.* **200** (1974) 1–32.
- [2] G. Bergman. Coproducts and some universal ring constructions *Trans. Amer. Math. Soc.* **200** (1974) 33–88.
- [3] J. J. Rotman. *Advanced modern algebra*. Prentice Hall, Saddle River NJ 2002.

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